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Vacuum Stress between Conducting Plates: An Image Solution*

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The zero-point fluctuations of the electromagnetic field give rise to an attractive force between two perfectly conducting parallel plates, the Casimir force. We discuss the structure of the electromagnetic stress-energy tensor in the region between the plates for finite temperatures as well as for the zero-temperature limit, and we describe the relationship of its components to the thermodynamic variables of the radiation field. The stress-energy tensor is defined so that infinite quantities never appear, and it is explicitly computed with the aid of an image-source construction of the Green's function. The finite-temperature case involves both an infinite set of spatial images and an infinite sum of temperature-dependent images.

1. INTRODUCTION

THE mutual electrical polarization of material bodies brought about by quantum-mechanical fluctuations results in an attractive force: At short distances this is the van der Waals interaction; at large distances the retarded propagation of the electromagnetic field becomes important. In this case, the force can be computed from the total energy of the system in interaction with the quantized electromagnetic field.[1] If the polarizability of the materials is extremely large, they behave as perfect conductors, and the interaction force can be calculated from the energy of the quantized electromagnetic field alone, which now exists only outside the bodies. In this spirit, Casimir[2] obtained the force between two parallel, perfectly conducting, infinite plates at zero temperature. In order to secure a finite result from the usual quadratically divergent expression for the vacuum fluctuation energy, it was necessary to introduce a strong convergence factor in the cavity-mode sum, to discard the contribution to the energy which is independent of the plate separation, and then to let the convergence factor approach unity.

We shall compute the complete electromagnetic stress-energy tensor of Casimir's problem for finite temperature[3] as well as for the zero-temperature limit. We define the stress-energy tensor in terms of a suitable limit of a bilinear field combination with finite spatial separation. This definition automatically removes the usual vacuum infinity, and we avoid the explicit manipulation of infinite quantities. The calculation of the stress-energy tensor is performed using an image-source construction of the electromagnetic field two-point Green's function. At zero temperature, this involves an infinite sequence of image sources displaced in space in a manner akin to the familiar electrostatic image solution of a point charge placed between two conducting plates. The Green's function for a finite-temperature ensemble can be represented by adding for each spatial image an infinite sum of temperature images displaced in imaginary time. In addition to calculating the stress-energy tensor, we discuss its structure in detail—in particular, the relationship of its components to the various thermodynamic variables of the radiation field.

The organization of this paper is perhaps unconventional: We present our major results in Sec. 2, deferring explicit calculations until Secs. 3 and 4. In Sec. 2, after properly defining the stress-energy tensor, we consider its structure at zero temperature. In this case, the conditions that the stress tensor be divergence-free and traceless, together with the simple geometry of the problem and the use of dimensional arguments, completely determine the whole tensor in terms of a single pure number. Using this result, we can directly verify the principle of virtual work—that the pressure on a conducting plate, as computed by the spatial stress tensor, agrees with the pressure implied by the variation of the energy with plate separation. We then generalize these considerations to the finite-temperature case. We show that the components of the stress tensor parallel to the surface of the plates are identical with the Helmholtz free energy per unit volume of the radiation field. By identifying the pressure on the plates with the spatial variation of the free energy at constant temperature, we obtain all the components of the stress-energy tensor in terms of a single function of a dimensionless variable: the temperature times the

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[1] H. B. G. Casimir and D. Polder, Phys. Rev. 73, 360 (1948). An alternative method, involving a randomly fluctuating classical electromagnetic field, has been used by E. M. Lifshitz, Zh. Eksperim. i Teor. Fiz. 29, 94 (1955) [English transl.: Soviet Phys.—JETP 2, 73 (1956)]; see also L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1960), Sec. 90; I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Usp. Fiz. Nauk 73, 381 (1961) [English transl.: Soviet Phys.—Usp. 4, 153 (1961)]. A recent discussion has been given by T. H. Boyer, Phys. Rev. 174, 1631 (1968).

[2] H. B. G. Casimir, Koninkl. Ned. Akad. Wetenschap. Proc. 51, 793 (1948).

[3] Finite-temperature calculations following the method of Casimir have been performed by M. Fierz, Helv. Phys. Acta 33, 855 (1960); F. Sauer, dissertation, Gottinger, 1962 (unpublished) (we have not had access to this work); J. Mehra, Physica 37, 145 (1967). Lifshitz (Ref. 1) has also calculated finite-temperature effects by his method. The works of Sauer and of Lifshitz have been critically compared by C. M. Hargreaves [Koninkl. Ned. Akad. Wetenschap. Proc. B68, 231 (1965)], who concludes that the results of Lifshitz are in error. The results of Sauer are in accord with the later work of Mehra and with those presented in this paper.

distance between the plates. We quote the value of this function, which is obtained in Sec. 4, and discuss the limiting cases of low and high temperatures. Finally, we discuss briefly the structure of the electromagnetic stress-energy tensor outside a perfectly conducting sphere at zero temperature.

In Sec. 3 we define the electromagnetic field two-point Green's function and then compute this function in terms of an infinite sum of image sources. The Green's function is generalized in Sec. 4 to a finite-temperature ensemble and is constructed in terms of a doubly infinite sum of images in both space and imaginary time.

2. NATURE OF THE STRESS

The quantum electrodynamic stress-energy tensor is, formally, the $\epsilon \to 0$ limit of the bilinear field combination[4]

$$T^{\mu\nu}(x,\epsilon)=F^{\mu\lambda}(x+\tfrac{1}{2}\epsilon)F^{\nu}{}_{\lambda}(x-\tfrac{1}{2}\epsilon) -\tfrac{1}{4}g^{\mu\nu}F^{\lambda\kappa}(x+\tfrac{1}{2}\epsilon)F_{\lambda\kappa}(x-\tfrac{1}{2}\epsilon). \quad (1)$$

This limit, in fact, does not exist, because its vacuum expectation value diverges as $(\epsilon^2)^{-2}$. Indeed, in an infinitely extended vacuum, the entire expectation value behaves as $(\epsilon^2)^{-2}$, reflecting the masslessness of the photon. We can achieve a finite and well-defined stress-energy operator by recognizing the homogeneous $(\epsilon^2)^{-2}$ character of this divergence, which we can therefore remove with the definition

$$T^{\mu\nu}(x)=\lim_{\epsilon\to 0}\left(1+\tfrac{1}{4}\epsilon^{\lambda}\frac{\partial}{\partial\epsilon^{\lambda}}\right)T^{\mu\nu}(x,\epsilon) \quad (2)$$

In our applications using the image-source construction of the Green's functions, this definition is tantamount to simply discarding that part of the Green's function corresponding to the true source—the infinitely extended vacuum contribution. Once this is done, it follows from the structure of the Green's function that there are no infinities in the stress tensor, even when it is evaluated at the surface of a plate.

Except at boundaries, the electromagnetic field is free, and thus the stress-energy tensor has a vanishing divergence

$$\partial_{\mu}T^{\mu\nu}(x)=0. \quad (3)$$

Since the photon is massless, the theory contains no intrinsic unit of length and is invariant under scale transformation of the electromagnetic field strength. This invariance is reflected in the vanishing of the trace of the stress-energy tensor[5]

$$T^{\mu}{}_{\mu}(x)=0. \quad (4)$$

We turn now to the nature of the stress between two perfectly conducting, parallel, infinite plates separated by a distance a. We orient the coordinate frame so that one plate is at $z=0$ while the other is at $z=a$. We shall have occasion to use the unit four-vector $\hat{z}^{\mu}=(0,0,0,1)$. The time axis is specified by $n^{\mu}=(1,0,0,0)$.

We will first consider the situation at zero temperature. In this case, the Green's function can be constructed with an infinite sequence of current-pulse image sources displaced along the z axis, but which exist at a common infinitesimal time duration. That there is no retardation in time between the various image sources is a result of the special symmetry of the parallel-plate geometry, which has pairs of sources at equal distances from a given plate so that no retardation is required for their radiation pulses to reach the plate simultaneously. Accordingly, the Green's function depends upon only the single four-vector $\hat{z}^{\mu}$, and the ground-state expectation value of the stress-energy tensor must be constructed from $\hat{z}^{\mu}\hat{z}^{\nu}$ and $g^{\mu\nu}$ with no additional reference to the time-axis normal n^{μ}. The traceless nature of the stress-energy tensor, together with the symmetry of the problem, requires that

$$\langle T^{\mu\nu}(x)\rangle_{(0)}=(\tfrac{1}{4}g^{\mu\nu}-\hat{z}^{\mu}\hat{z}^{\nu})f(z). \quad (5)$$

The function $f(z)$ must, in fact, be constant to make the stress-energy tensor free of divergence. Its dimension, energy per unit volume, gives the final structure

$$\langle T^{\mu\nu}\rangle_{(0)}=(\tfrac{1}{4}g^{\mu\nu}-\hat{z}^{\mu}\hat{z}^{\nu})(\hbar c/a^4)\gamma, \quad (6)$$

in which γ is a pure number. The explicit construction of the Green's function in Sec. 3 gives, of course, precisely this result and supplies the numerical value

$$\gamma=\frac{1}{2\pi^2}\sum_{l=1}^{\infty}l^{-4}=(1/2\pi^2)\zeta(4)=\pi^2/180 \quad (7)$$

The energy density between the plates

$$\langle T^{00}\rangle_{(0)}=-\tfrac{1}{4}(\hbar c/a^4)\gamma=-(\pi^2/720)(\hbar c/a^4) \quad (8)$$

and pressure on one of the plates

$$\langle T^{33}\rangle_{(0)}=-\tfrac{3}{4}(\hbar c/a^4)\gamma=-(\pi^2/240)(\hbar c/a^4) \quad (9)$$

agree with the results obtained by Casimir.[2] In particular, the pressure on a plate is negative, corresponding to an attractive force. It follows from the general structure of the stress-energy tensor (6) that the calculation of this pressure from the spatial stress agrees

[4] Our metric has the signature $(-1,1,1,1)$. In Secs. 3 and 4 we use natural units with Planck's constant, the Boltzmann constant, and the velocity of light unity, $\hbar=k=c=1$.

[5] The direct connection of zero mass and the vanishing of the trace of the stress tensor is not a general result. For example, the trace of a spin-zero, massless meson field stress tensor

$$T^{\mu\nu}=\partial^{\mu}\phi\partial^{\nu}\phi-g^{\mu\nu}\tfrac{1}{2}\partial^{\lambda}\phi\partial_{\lambda}\phi$$

does not vanish. The connection depends upon the special coordinate transformation character of the electromagnetic field, where the stress tensor can be identified by the response of the Lagrange function to a general coordinate variation with the field strength transforming as a contravariant, rank-2 tensor density. In this special circumstance, a scale transformation is a particular case of a general coordinate transformation, and thus the response is the trace of the stress tensor.

with the value obtained from the principal of virtual work, where it is defined by the variation of the energy per unit area induced by a change in the plate separation:

$$-\frac{\partial}{\partial a}[a\langle T^{00}\rangle_{(0)}]=\frac{\partial}{\partial a}[\tfrac{1}{3}(\hbar c/a^3)\gamma]=\langle T^{33}\rangle_{(0)}. \quad (10)$$

The components of the stress along a plate, $\langle T^{11}\rangle_{(0)}$ and $\langle T^{22}\rangle_{(0)}$, are also in accord with the principle of virtual work. The system is not altered significantly if a perfectly conducting wall is erected between the plates at a great distance from the origin. If such a wall, whose normal points along either the x or the y axis, is moved a distance δd, the energy per unit area changes by an amount $\delta d\langle T^{00}\rangle_{(0)}$, which is precisely the negative of the product of the stress and the spatial displacement, since according to the general form (6),

$$\langle T^{00}\rangle_{(0)}=-\langle T^{11}\rangle_{(0)}=-\langle T^{22}\rangle_{(0)}. \quad (11)$$

We turn now to the situation at a finite temperature T. In the limit of a very large separation between the plates only blackbody radiation appears. The expectation value of the stress in the canonical ensemble must become uniform and isotopic, and the stress-energy tensor can depend upon only the time-axis normal n^μ. The constraint of vanishing trace, together with the dimensionality of this tensor, now gives the limiting structure

$$\langle T^{\mu\nu}\rangle_{(T)}{}^{(\infty)}=(g^{\mu\nu}+4n^\mu n^\nu)(kT/\hbar c)^3kT\sigma, \quad (12)$$

with σ a pure number. The image construction in imaginary time of the temperature-dependent Green's function of Sec. 4 automatically gives this structure and the explicit value

$$\sigma=\frac{2}{\pi^2}\sum_{l=1}^{\infty}l^{-4}=\pi^2/45. \quad (13)$$

Thus, we reproduce the well-known result[6] that the energy density and pressure of a photon gas of infinite volume are given by

$$\langle T^{00}\rangle_{(T)}{}^{(\infty)}=(\pi^2/15)(kT/\hbar c)^3kT \quad (14)$$

and

$$\langle T^{kl}\rangle_{(T)}{}^{(\infty)}=\delta^{kl}\tfrac{1}{3}\langle T^{00}\rangle_{(T)}{}^{(\infty)}. \quad (15)$$

We may take account of the limits (6) and (12) and write the stress-energy tensor at finite temperature and plate separation as

$$\langle T^{\mu\nu}\rangle_{(T)}=\langle T^{\mu\nu}\rangle_{(0)}+\langle T^{\mu\nu}\rangle_{(T)}{}^{(\infty)}+\langle T^{\mu\nu}\rangle_{(T)}{}^{(a)}, \quad (16)$$

so that we need consider only the correction $\langle T^{\mu\nu}\rangle_{(T)}{}^{(a)}$, which vanishes at zero temperature and at large plate separation. We can obtain the structure of this correction if we make use of the qualitative result of the Green's-function construction of Sec. 4 that the stress is uniform between the plates. Then the now familiar conditions of vanishing trace and proper dimensionality give

$$\langle T^{\mu\nu}\rangle_{(T)}{}^{(a)}=(n^\mu n^\nu+\hat{z}^\mu\hat{z}^\nu)(kT/a^3)s(\xi)+(4\hat{z}^\mu\hat{z}^\nu-g^{\mu\nu})(\hbar c/a^4)f(\xi), \quad (17)$$

in which s and f are arbitrary functions of the dimensionless variable

$$\xi=kTa/\hbar c. \quad (18)$$

In general, a given component of the stress is given by the appropriate spatial derivative of the Helmholtz free energy at constant temperature. Now, in analogy to our previous discussion of the principle of virtual work at zero temperature, the components of the stress parallel to the plates,

$$\langle T^{11}\rangle_{(T)}{}^{(a)}=\langle T^{22}\rangle_{(T)}{}^{(a)}=-(\hbar c/a^4)f(\xi), \quad (19)$$

must be identified with the negative of a Helmholtz free-energy density. With this identification of $(\hbar c/a^4)f(\xi)$ as the correction to Helmholtz free-energy density for finite plate separation and temperature, we can compute the pressure correction on one of the plates by the partial derivative at constant temperature,

$$\langle T^{33}\rangle_{(T)}{}^{(a)}=-\frac{\partial}{\partial a}[(\hbar c/a^3)f(\xi)], \quad (20a)$$

as well as by the general form (17),

$$\langle T^{33}\rangle_{(T)}{}^{(a)}=(kT/a^3)s(\xi)+3(\hbar c/a^4)f(\xi), \quad (20b)$$

and hence we must have

$$s(\xi)=-\frac{d}{d\xi}f(\xi). \quad (21)$$

The image solution of Sec. 4 is in accord with these results and provides the explicit functional form

$$f(\xi)=-\frac{1}{4\pi^2}\sum_{l,m=1}^{\infty}\frac{(2\xi)^4}{[l^2+(2\xi)^2m^2]^2} \quad (22)$$

It follows from the form (17) that the energy-density correction is given by

$$\langle T^{00}\rangle_{(T)}{}^{(a)}=(\hbar c/a^4)u(\xi), \quad (23)$$

with

$$u(\xi)=f(\xi)+\xi s(\xi), \quad (24)$$

which corresponds to the thermodynamic connection between the internal energy U, the Helmholtz free energy F, and the entropy S:

$$U=F+TS. \quad (25)$$

Accordingly, we find that $ks(\xi)$ is the correction to the entropy per unit volume of the radiation field, and we have related all the components of the stress-energy tensor to thermodynamic variables. Since the photon is massless, photon number is not conserved, the

[6] M. Planck, Verhandl. Deut. Physikalischen Ges. 2, 237 (1900).

chemical potential of the radiation field vanishes, and the thermodynamic energy balance is given by

$$dU-TdS+dW=0, \tag{26}$$

in which dW is the external work contribution. We have, by construction, satisfied this energy-balance equation at constant temperature, and we need only require that it hold at constant volume,

$$\frac{\partial U}{\partial T}=T\frac{\partial S}{\partial T}, \tag{27a}$$

or, in terms of the Helmholtz free energy,

$$S=-\partial F/\partial T \tag{27b}$$

This relation between the entropy and the free energy is precisely the relation between the dimensionless functions $s(\xi)$ and $f(\xi)$ exhibited in Eq. (21). This correspondence is a direct consequence of the zero mass of the photon, which requires on the one hand that the stress-energy tensor be traceless and, on the other, that the chemical potential vanish. The thermodynamic relationships which we have exhibited for the correction term $\langle T^{\mu\nu}\rangle_{(T)}{}^{(a)}$ hold for the complete stress tensor, since they are trivially satisfied by its other pieces.

The double-sum representation (22) of the free-energy function $f(\xi)$ shows that this function is singular at all rational points along the imaginary axis in the complex ξ plane. This natural boundary corresponds to the divergence of the partition function which occurs when the temperature is analytically continued to a pure imaginary value. The double-sum representation also exhibits the inversion symmetry

$$f(1/4\xi)=(2\xi)^{-4}f(\xi). \tag{28}$$

Accordingly, the knowledge of one asymptotic form determines the behavior of the function for both large and small ξ. We can obtain the large-ξ behavior if we use a sum formula which is easily obtained by the familiar contour integration method:

$$\sum_{m=1}^{\infty}\frac{1}{[(l/2\xi)^2+m^2]^2}=\frac{2\pi\xi^3}{l^3}\frac{\cosh(\pi l/2\xi)}{\sinh(\pi l/2\xi)}+\frac{(\pi\xi)^2}{l^2}\sinh^{-2}\left(\frac{\pi l}{2\xi}\right)-\frac{8\xi^4}{l^4}. \tag{29}$$

Thus, for $\xi\rightarrow 0$,

$$\begin{aligned}f(\xi)&=-(\xi^3/2\pi)\sum_{l=1}^{\infty}l^{-3}+(2\xi^4/\pi^2)\sum_{l=1}^{\infty}l^{-4}-[(\xi^3/\pi)+\xi^2]e^{-(\pi/\xi)}\\&=-(\xi^3/2\pi)\zeta(3)+(\xi^4\pi^2/45)-[(\xi^3/\pi)+\xi^2]e^{-(\pi/\xi)}+O(e^{-(2\pi/\xi)}),\end{aligned} \tag{30}$$

in which $\zeta(3)$ has the numerical value 1.202 For $\xi\rightarrow\infty$,

$$f(\xi)=-(\xi/8\pi)\zeta(3)+(\pi^2/720)-[(\xi/4\pi)+\xi^2]e^{-4\pi\xi}+O(e^{-8\pi\xi}). \tag{31}$$

The limit $\xi=kTa/hc\rightarrow 0$ corresponds physically to small temperature or small plate separation. In this case we find that, neglecting exponentially small terms, for $Ta\rightarrow 0$,

$$\langle T^{00}\rangle_{(T)}{}^{(a)}=[\zeta(3)/\pi^2](kT/a)(kT/hc)^2-(\pi^2/15)kT(kT/hc)^3 \tag{32}$$

and

$$\langle T^{33}\rangle_{(T)}{}^{(a)}=-(\pi^2/45)(kT/hc)^3(kT). \tag{33}$$

The corresponding behavior of the complete energy density and the pressure on a plane is, for $Ta\rightarrow 0$,

$$\langle T^{00}\rangle_{(T)}=-(\pi^2/720)(hc/a^4)+[\zeta(3)/\pi^2](kT/a)(kT/hc)^2 \tag{34}$$

and

$$\langle T^{33}\rangle_{(T)}=-(\pi^2/240)(hc/a^4). \tag{35}$$

Note that the terms which are independent of the plate separation have all cancelled. The absence of such black-body radiation terms happens because at low temperatures or, equivalently, at small plate separation no modes of the radiation field propagating normally to the plates can be excited. The modes propagating along the plates contribute to the energy density, as seen in Eq. (34), but do not contribute to the pressure on a plate, Eq. (35), since the total free energy in these modes is independent of the plate separation. The corrections to these limits[7] are exponentially small in the parameter $\pi hc/kTa$.

In the high-temperature or large-plate-separation limit, we have, neglecting exponentially small terms, for $Ta\rightarrow\infty$,

$$\langle T^{00}\rangle_{(T)}{}^{(a)}=(\pi^2/720)(hc/a)^4 \tag{36}$$

[7] Since the force becomes exceedingly small at large plate separation, only this $Ta\rightarrow 0$ limit is measurable. In addition to the pressure discussed in the text, which is due exclusively to the radiation between the plates, in any experiment there is also a pressure from the blackbody radiation external to the plates which cancels the blackbody contribution to the interior pressure. If we neglect exponentially small terms, the expression for the total experimentally observed pressure is the sum of Eqs. (35) and (15). At room temperature this pressure in dyn/cm² is $-p=0.01300a^{-4}+2\times10^{-6}$, where a is measured in microns. An experimental measurement of the vacuum stress between conducting plates has been made by M. J. Sparnaay, Physica 24, 751 (1958). In addition, there have been a number of similar measurements with dielectric plates [for experimental results and references, see W. Block, J. G. V. de Jongh, J. Th.G. Overbeek, and M. J. Sparnaay, Trans. Faraday Soc. 56, 1597 (1960)], for which the theory must be modified somewhat (Lifshitz *et al.*, Ref. 1). The experimental results are consistent, in general, with the existence of attractive forces, which vary inversely with the plate separation but are of insufficient accuracy to clearly verify the exponent and coefficient of the leading term for the pressure.

$$\langle T^{33}\rangle_{(T)}{}^{(a)} = -[\zeta(3)/4\pi](kT/a^3) + (\pi^2/240)(\hbar c/a^4). \quad (37)$$

The corresponding limits of the total energy density and pressure are given for $Ta\to\infty$ by

$$\langle T^{00}\rangle_{(T)} = (\pi^2/15)(kT/\hbar c)^3 kT \quad (38)$$

and

$$\langle T^{33}\rangle_{(T)} = (\pi^2/45)(kT/\hbar c)^3 kT - [\zeta(3)/4\pi](kT/a^3). \quad (39)$$

Note that in this case all the terms independent of the temperature have cancelled—no contributions occur with the character of the zero-point vacuum fluctuations. The energy density is that of blackbody radiation, while the pressure on a plate contains a purely classical term (the physical significance of which is unclear) in addition to the blackbody radiation contribution. The corrections to these limits are exponentially small in the parameter $4\pi kTa/\hbar c$.

The correction to the pressure on a plate may be written as

$$\langle T^{33}\rangle_{(T)}{}^{(a)} = (\hbar c/a^4)p(\xi), \quad (40)$$

in which

$$p(\xi) = -\xi^4(d/d\xi)[\xi^{-3}f(\xi)]. \quad (41)$$

We can make use of the sum formula (29) and the double-sum representation (22) to express this as

$$p(\xi) = \frac{\xi}{2\pi}\left(\frac{d}{d\xi^{-1}}\right)^2 \sum_{l=1}^{\infty} l^{-3}\left[\frac{1+e^{-\pi l/\xi}}{1-e^{-\pi l/\xi}} - \frac{2\xi}{l\pi}\right]. \quad (42)$$

On expanding the denominator in this formula and performing the l-sum, we get

$$p(\xi) = -\pi\xi\sum_{n=1}^{\infty} n^2 \ln(1-e^{-n\pi/\xi}) - (\pi^2/45)\xi^4, \quad (43)$$

which is precisely the pressure correction obtained by Mehra.[3]

It is interesting to compare our results on the stress-energy tensor for conducting plates with the structure of the zero-temperature stress-energy tensor in the region outside a perfectly conducting sphere[8] of radius a centered at the origin. The symmetry of this problem, coupled with dimensional considerations and the condition that the stress tensor be divergence-free, requires that the spatial stress have the form

$$\langle T^{kl}\rangle_{(0)} = (\delta^{kl} - \hat{r}^k\hat{r}^l)\tfrac{1}{2}r\frac{\partial}{\partial r}(\hbar c/ar^3)f(r/a) + \delta^{kl}(\hbar c/ar^3)f(r/a). \quad (44)$$

The vanishing of the trace of the complete stress-energy tensor then determines the energy density

$$\langle T^{00}\rangle = (\hbar c/ar^2)\frac{\partial}{\partial r}f(r/a) \quad (45)$$

The total energy in the vacuum fluctuations is

$$U = 4\pi\int_a^\infty r^2 dr \langle T^{00}\rangle_{(0)} = -4\pi(\hbar c/a)f(1). \quad (46)$$

Accordingly, we find that the pressure on the sphere as given by the principle of virtual work,

$$P = \frac{1}{4\pi a^2}\frac{\partial}{\partial a}U = (\hbar c/a^4)f(1), \quad (47a)$$

agrees precisely with that obtained from the normal component of the stress tensor evaluated on the surface of the sphere

$$P = \hat{r}_k\langle T^{kl}\rangle_{(0)}\hat{r}_l|_{r=a} = (\hbar c/a^4)f(1). \quad (47b)$$

The calculation of the function $f(r/a)$ appears to be a difficult one. An image construction cannot be employed here because of the retarded propagation character of a radiation pulse. It appears necessary to use a decomposition into spherical wave modes, and it seems unlikely that the partial-wave sum can be put into closed form. An approximate evaluation has been made by Boyer,[8] who found a negative pressure with

$$f(1) \simeq -(0.09/8\pi).$$

3. ZERO-TEMPERATURE GREEN'S FUNCTION

The free electromagnetic field-strength tensor $F^{\mu\nu}(x)$ is characterized by the field equations

$$\partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} = 0, \quad (48)$$

$$\partial_\mu F^{\mu\nu} = 0, \quad (49)$$

and the nonvanishing equal-time commutator

$$i[F^{0k}(\mathbf{r},t), F^{lm}(\mathbf{r}',t)] = (\delta^{km}\partial^l - \delta^{kl}\partial^m)\delta(\mathbf{r}-\mathbf{r}'). \quad (50)$$

Since the equal-time commutator of the field strengths involves a derivative of the δ function, their time-ordered product is not covariant[9] under Lorentz transformations. A covariant time-ordered product T^* can be obtained by adjoining an appropriate contact or "seagull" term to the ordinary time-ordered product T:

$$iT^*(F^{\mu\nu}(x)F^{\lambda\kappa}(x')) = iT(F^{\mu\nu}(x)F^{\lambda\kappa}(x')) + (g^{\mu\kappa}n^\nu n^\lambda - g^{\mu\lambda}n^\nu n^\kappa + g^{\nu\lambda}n^\mu n^\kappa - g^{\nu\kappa}n^\mu n^\lambda)\delta(x-x'). \quad (51)$$

It follows from the field equations and commutation relations that this covariant time-ordered product satisfies

$$\partial^\sigma iT^*(F^{\mu\nu}(x)F^{\lambda\kappa}(x')) + \text{perms.} = 0 \quad (52)$$

[8] This problem has been considered by H. B. G. Casimir [Physica 19, 846 (1953)] and more recently it has been discussed at some length by T. H. Boyer, Phys. Rev. 174, 1764 (1968).

[9] L. S. Brown, Phys. Rev. 150, 1338 (1966).

and

$$\partial_\mu iT^*(F^{\mu\nu}(x)F^{\lambda\kappa}(x'))=(g^{\nu\kappa}\partial^\lambda-g^{\nu\lambda}\partial^\kappa)\delta(x-x'). \quad (53)$$

The expectation value of the covariant time-ordered product in the infinite vacuum

$$D_+{}^{\mu\nu;\lambda\kappa}(x-x')=\langle iT^*(F^{\mu\nu}(x)F^{\lambda\kappa}(x'))\rangle_{(0)}{}^{(\infty)} \quad (54)$$

is defined by the field equations (52) and (53) and by the usual positive-frequency boundary condition. The first field equation (52) is satisfied if we write this free-space Green's function in terms of a curl:

$$D_+{}^{\mu\nu;\lambda\kappa}(x-x')=d^{\mu\nu;\lambda\kappa}D_+(x-x'), \quad (55)$$

$$d^{\mu\nu;\lambda\kappa}=\partial^\mu\partial'^\lambda g^{\nu\kappa}-\partial^\nu\partial'^\lambda g^{\mu\kappa}+\partial^\nu\partial'^\kappa g^{\mu\lambda}-\partial^\mu\partial'^\kappa g^{\nu\lambda}. \quad (56)$$

Since

$$\partial'^\lambda D_+(x-x')=-\partial^\lambda D_+(x-x'), \quad (57)$$

the second field equation, (53), requires that

$$-\partial^2 D_+(x-x')=\delta(x-x'), \quad (58)$$

which has the familiar solution satisfying the positive-frequency boundary conditions

$$D_+(x)=i\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2|k|}e^{i\mathbf{k}\cdot\mathbf{r}-i|k||t|}=\frac{i}{4\pi^2}\frac{1}{x^2+i\epsilon} \quad (59)$$

We can now consider the case of a single, perfectly conducting, infinite plate placed at $z=0$. The tangential components of the electric field and the normal component of the magnetic field must vanish on the conductor:

$$F^{01}=F^{02}=F^{12}=0 \text{ on the plate}. \quad (60)$$

The Green's function $W_+{}^{\mu\nu;\lambda\kappa}(x,x')$ for this problem is easily obtained by adding an image function to the free-space solution. This image function is the free-space Green's function for an image source placed at the reflected coordinate

$$\bar{x}'^\mu=(x'^0,x'^1,x'^2,-x'^3). \quad (61)$$

To satisfy the boundary conditions (60), the image current components parallel to the plate and the image charge density sign must be inverted relative to the corresponding source currents and charge densities. Hence, we define

$$\bar{g}^{\mu\nu}=g^{\mu\nu}-2\hat{z}^\mu\hat{z}^\nu=\begin{pmatrix}-1&&&\\&+1&&\\&&+&\\&&&-1\end{pmatrix} \quad (62)$$

Accordingly, we write

$$W_+{}^{\mu\nu;\lambda\kappa}(x,x')=d^{\mu\nu;\lambda\kappa}D_+(x-x')-\bar{d}^{\mu\nu;\lambda\kappa}D_+(x-\bar{x}'), \quad (63)$$

where

$$\bar{d}^{\mu\nu;\lambda\kappa}=\partial^\mu\partial'^\lambda\bar{g}^{\nu\kappa}-\partial^\nu\partial'^\lambda\bar{g}^{\mu\kappa}+\partial^\nu\partial'^\kappa\bar{g}^{\mu\lambda}-\partial^\mu\partial'^\kappa\bar{g}^{\nu\lambda}. \quad (64)$$

Since the image function is expressed as a curl, the first of the field equations, (52), is obeyed; since

$$\partial'^\lambda D_+(x-\bar{x}')=-\bar{g}^{\lambda\nu}\partial_\nu D_+(x-\bar{x}'), \quad (65)$$

it is a simple matter to show that the second field equation, (53), holds as well. Finally, it is straightforward to verify directly that the boundary conditions (60) are satisfied, while the positive-frequency boundary conditions in time obviously hold. The simplicity of this solution results from the simplicity of the geometry. In particular, an image technique analogous to that of electrostatics is applicable to the radiation field because the retardation times of the source radiation pulse and the image pulse are identical.

The generalization of this Green's function to the situation where there are two infinite, perfectly conducting plates, one at $z=0$ the other at $z=a$, is immediate. In this case we use an infinite sequence of image sources of alternating types displaced along the z axis to secure the solution[10]

$$\begin{aligned}G_+{}^{\mu\nu;\lambda\kappa}(x,x')&=\langle iT^*(F^{\mu\nu}(x)F^{\lambda\kappa}(x'))\rangle_{(0)}\\&=d^{\mu\nu;\lambda\kappa}\sum_{l=-\infty}^{\infty}D_+(x-x'-2al\hat{z})\\&\quad-\bar{d}^{\mu\nu;\lambda\kappa}\sum_{l=-\infty}^{\infty}D_+(x-\bar{x}'-2al\hat{z}).\end{aligned} \quad (66)$$

Each term in the sum corresponds to a particular reflection of the original source pulse by one of the plates. Since an infinite number of such reflections is possible, our sums contain an infinite number of image terms. The stress-energy tensor appears as

$$\begin{aligned}\langle T^{\mu\nu}\rangle&=(-i)G^{\mu\lambda;\nu}{}_\lambda(x,x)-\tfrac{1}{4}g^{\mu\nu}(-i)G^{\lambda\kappa}{}_{;\lambda\kappa}(x,x)\\&=-\partial^\mu\partial^\nu\sum_{l=-\infty}^{\infty}{}'(-i)D_+(x-x'-2al\hat{z})\big|_{x=x'},\end{aligned} \quad (67)$$

where, according to the definition (2), it is implicit that the vacuum contribution to the Green's function is omitted, so that the value $l=0$ is excluded from the sum. Using the explicit functional form of the zero-mass propagator (59), we easily derive the results quoted in Sec. 2, Eqs. (6) and (7).

4. FINITE-TEMPERATURE GREEN'S FUNCTION

At finite temperatures, we must use the canonical ensemble average[11]

$$\langle X\rangle_{(T)}=(\mathrm{Tr}e^{-\beta H})^{-1}\,\mathrm{Tr}e^{-\beta H}X, \quad (68)$$

[10] The sums which occur here can be done by the usual contour integration method to put the solution in a closed form involving hyperbolic cosines. There is, however, no necessity to do this for our purposes.

[11] Our treatment is a standard one. See, for example, L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

in which β is the inverse temperature $\beta=T^{-1}$, and H is the Hamiltonian of the system, which also governs its time development by

$$X(t)=e^{iHt}X(0)e^{-iHt}. \tag{69}$$

The cyclic invariance of the trace

$$\mathrm{Tr}XY=\mathrm{Tr}YX \tag{70}$$

and the Heisenberg equation of motion (69) imply that time-dependent correlations of the form

$$\langle A(t)B(t')\rangle_{(T)}$$

depend only upon the time difference $t-t'$ and, in addition, satisfy the symmetry

$$\langle A(t)B(t')\rangle_{(T)}=\langle B(t')A(t+i\beta)\rangle_{(T)}. \tag{71}$$

Accordingly, if we write the various orderings of the correlations as Fourier integrals,

$$\langle A(t)B(t')\rangle_{(T)}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}g^{(+)}(\omega)e^{-i\omega(t-t')}, \tag{72a}$$

$$\langle B(t')A(t)\rangle_{(T)}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}g^{(-)}(\omega)e^{-i\omega(t-t')}, \tag{72b}$$

$$\langle[A(t),B(t')]\rangle_{(T)}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}c(\omega)e^{-i\omega(t-t')}, \tag{72c}$$

we have the connection

$$g^{(\pm)}(\omega)=\pm c(\omega)(1-e^{\pm\beta\omega})^{-1}. \tag{73}$$

In particular, the time-ordered product is determined by the value of the commutator.

The commutator of the free electromagnetic field strengths is a numerical quantity. Hence, its value in the canonical ensemble average is the same as its ground-state expectation value. Furthermore, for $t>t'$, the commutator is simply twice the real part of the Green's function, from which its value for $t<t'$ can be obtained by analytic continuation. Accordingly, for the case of infinitely extended space, we have

$$\begin{aligned}D^{\mu\nu;\lambda\kappa}(x-x')&=\langle i[F^{\mu\nu}(x),F^{\lambda\kappa}(x')]\rangle_{(T)}{}^{(\infty)}\\&=d^{\mu\nu;\lambda\kappa}D(x-x'),\end{aligned} \tag{74}$$

with

$$D(x)=i\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}e^{i\mathbf{k}\cdot\mathbf{r}}[e^{-i|k|t}-e^{+i|k|t}]. \tag{75}$$

We can now readily obtain the finite-temperature Green's function for this case of an infinitely extended space by using the connection (73) between the two operator orderings and the commutator. The non-covariant "seagull" contributions to the covariant time-ordered product (51) are canceled by the δ-function terms which arise when the step functions $\theta(t-t')$ and $\theta(t'-t)$ that define the two time orderings are commuted through the differential operator $d^{\mu\nu;\lambda\kappa}$, and we obtain

$$\begin{aligned}D_T{}^{\mu\nu;\lambda\kappa}(x-x')&=\langle iT^*(F^{\mu\nu}(x)F^{\lambda\kappa}(x'))\rangle_{(T)}{}^{(\infty)}\\&=d^{\mu\nu;\lambda\kappa}D_T(x-x'),\end{aligned} \tag{76}$$

in which

$$\begin{aligned}D_T(x)=i\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}e^{i\mathbf{k}\cdot\mathbf{r}}\Bigg[\theta(t)\left(\frac{e^{-i|k|t}}{1-e^{-\beta|k|}}-\frac{e^{+i|k|t}}{1-e^{+\beta|k|}}\right)\\-\theta(-t)\left(\frac{e^{-i|k|t}}{1-e^{+\beta|k|}}-\frac{e^{+i|k|t}}{1-e^{-\beta|k|}}\right)\Bigg].\end{aligned} \tag{77}$$

The various terms which occur here may be combined in the form

$$D_T(x)=D_+(x)+D_T'(x), \tag{78}$$

where $D_+(x)$ is the vacuum zero-mass propagator (59) and

$$\begin{aligned}D_T'(x)=i\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-\beta|k|}(1-e^{-\beta|k|})^{-1}\\\times(e^{-i|k|t}+e^{+i|k|t}).\end{aligned} \tag{79}$$

The exponential damping factors in the integral (79) ensure that the integral is well defined. We can therefore expand the temperature-dependent denominator to put this correction function in the form of an infinite sum of temperature images in imaginary time. The images are displaced in time by $im\beta$, where m is any positive or negative integer excluding zero. The term with $m=0$ is simply the vacuum propagator $D_+(x)$, so that the complete propagator Eq. (78) becomes

$$D_T(x)=\sum_{m=-\infty}^{\infty}D_+(x^\mu-im\beta n^\mu). \tag{80}$$

Using this form, the temperature correction to the stress-energy tensor in free space is easily computed, obtaining the results quoted in Eqs. (12) and (13). Our definition of the stress-energy tensor [Eq. (2)] excludes the $m=0$ vacuum propagator term.

The extension of this formulation to the finite-temperature radiation field between two perfectly conducting infinite plates is direct: The presence of the plates is accounted for by an infinite sum of spatial image functions, and we have

$$\begin{aligned}G_T{}^{\mu\nu;\lambda\kappa}&(x,x')\\&=\langle iT^*(F^{\mu\nu}(x)F^{\lambda k}(x'))\rangle_{(T)}\\&=d^{\mu\nu;\lambda\kappa}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}D_+(x-x'-2al\hat{z}-im\beta n)\\&\quad-\bar{d}^{\mu\nu;\lambda\kappa}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}D_+(x-\bar{x}'-2al\hat{z}-im\beta n).\end{aligned} \tag{81}$$

It is a simple matter to verify that the second double sum involving $\bar{d}^{\mu\nu;\lambda\kappa}$ does not contribute to the stress-energy tensor. The $l=m=0$ term of the first double sum is excluded, of course, by definition (2). The part of the first sum, with $m=0$, $l\neq 0$, gives the zero-temperature contribution to the stress-energy tensor which we have already considered; the part with $l=0$, $m\neq 0$ gives the blackbody contribution which we have just discussed. It is straightforward to show that the sum remaining with neither l nor m vanishing gives the finite-temperature, finite-plate separation correction quoted in Eqs. (17)–(22).

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