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G. Jordan Maclay & Eric W. Davis

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Testing a Quantum Inequality with a Meta-analysis of Data for Squeezed Light

G. Jordan Maclay[1] · Eric W. Davis[2,3]

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Abstract

In quantum field theory, coherent states can be created that have negative energy density, meaning it is below that of empty space, the free quantum vacuum. If no restrictions existed regarding the concentration and permanence of negative energy regions, it might, for example, be possible to produce exotic phenomena such as Lorentzian traversable wormholes, warp drives, time machines, violations of the second law of thermodynamics, and naked singularities. Quantum Inequalities (QIs) have been proposed that restrict the size and duration of the regions of negative quantum vacuum energy that can be accessed by observers. However, QIs generally are derived for situations in cosmology and are very difficult to test. Direct measurement of vacuum energy is difficult and to date no QI has been tested experimentally. We test a proposed QI for squeezed light by a meta-analysis of published data obtained from experiments with optical parametric amplifiers and balanced homodyne detection. Over the last three decades, researchers in quantum optics have been trying to maximize the squeezing of the quantum vacuum and have succeeded in reducing the variance in the quantum vacuum fluctuations to − 15 dB. To apply the QI, a time sampling function is required. In our meta-analysis different time sampling functions for the QI were examined, but in all physically reasonable cases the QI is violated by much or all of the measured data. This brings into question the basis for QI. Possible explanations are given for this surprising result.

Keywords Squeezed light · Quantum inequality · Vacuum energy · Vacuum fluctuations · Negative energy · Optical parametric amplifier

✉ G. Jordan Maclay
jordanmaclay@quantumfields.com

Eric W. Davis
EWDavis@earthtech.org

1 Quantum Fields LLC, St. Charles, IL 60174, USA

2 Institute for Advanced Studies at Austin, 11855 Research Blvd., Austin, TX 78759, USA

3 Early Universe, Cosmology and Strings Group, Center for Astrophysics, Space Physics and Engineering Research, Baylor University, Waco, TX 76798, USA

1 Introduction

In quantum field theory, the vacuum expectation value of the normally ordered or renormalized energy density $\langle T_{oo} \rangle$ need not be positive. For example, a superposition of a vacuum state (n = 0) and a two photon state (n = 2), can have negative renormalized energy density with the proper choice of coefficients. Squeezed light can have a negative energy density. From theory and experiment, we know that static negative energy densities associated with vacuum states are concentrated in narrow spatial regions, e. g., inside a parallel plate Casimir cavity with small plate separation or in the region near the Schwarzschild radius in the Boulware vacuum where the energy density is everywhere negative as seen by static observers. There is no known way to directly measure vacuum energy density. On the other hand, the total energy of a system is believed to always be positive or zero. For example, the sum of the mass energy of the plates plus the negative vacuum energy inside the cavity is positive [1,2]. The classical energy conditions imply that an inertial observer who initially encounters some negative energy density must encounter compensating positive energy density at some arbitrary time in the future. Quantum Inequalities (QIs) have been derived for the free vacuum quantum electromagnetic field, with no sources or boundaries, which constrain the magnitude and duration of negative energy densities relative to the energy density of an underlying reference vacuum state. The QI places bounds on quantum violations of the classical energy conditions [3,4]. The QI is formulated as a mathematical bound on the average of the quantum expectation value of a free field's energy-momentum tensor in the vacuum state, where the average is taken along an observer's timelike or null worldline using time sampling functions. Contrary to the classical energy conditions, the QI dictates that the more negative the energy density is in some time interval T, the shorter the duration T of the interval, so that an inertial observer cannot encounter arbitrarily large negative energy densities that last for arbitrarily long time intervals. An inertial observer must encounter compensating positive energy density no later than after a time T, which is inversely proportional to the magnitude of the initial negative energy density.

In QI, restrictions are placed on the integral of the vacuum expectation value of the renormalized energy density $\langle T_{oo} \rangle$ multiplied by a sampling function. For the electromagnetic field in flat space-time, with a normalized time sampling function of $f(t) = (t_o/\pi)(1/(t^2 + t_o^2))$, Ford has shown [5]

$$\hat{\rho} \equiv \frac{t_o}{\pi} \int_{-\infty}^{+\infty} \frac{\langle T_{oo} \rangle}{t^2 + t_o^2} dt \geqslant -\frac{3}{16\pi^2} \frac{\hbar c}{(ct_o)^4} \tag{1}$$

To give a frame of reference, this can be compared to the vacuum energy density within an ideal parallel plate Casimir cavity of separation a:

$$\langle T_{oo} \rangle_{Cas} = -\frac{\pi^2}{720} \frac{\hbar c}{a^4} \tag{2}$$

The ratio of the numerical factors for the free field to the Casimir cavity is 1.4, so a negative energy density $\hat{\rho}$ equal to that in a perfectly conducting parallel plate cavity

of spacing a can exist no longer than for a time $t_o \sim a/c$, about 3×10^{-16} s for a typical experiment. As the sampling time t_o increases, $\hat{\rho}$ rapidly goes to zero. (Note however, that as derived, the QIs do not apply directly to the Casimir cavity since it has boundaries. Also, to test Eq. 1 experimentally, one must make an absolute measurement of the vacuum energy density, an experimental challenge for which no solution has yet been found. Some progress is due to Riek et al. who were able to directly probe the spectrum of squeezed vacuum fluctuations of the electric field in the multi-THz range using femtosecond laser pulses [6].)

If the laws of quantum field theory placed no restrictions on negative energy, then it might be possible to produce surprising macroscopic effects such as violations of the second law of thermodynamics, traversable wormholes, warp drives, and possibly time machines [5]. QI appear to restrict these violations of the second law [7,8].

A quantum inequality has been derived for squeezed light by Marecki [9]. Squeezed light has a nonclassical distribution of the quadrature components (typically phase and amplitude), which may be considered as the canonical momentum and position components of an equivalent harmonic oscillator corresponding to the frequency of the electromagnetic radiation being considered. Squeezed states are routinely made in quantum optics experiments in the process of parametric down conversion, in which an incident photon is converted in a non-linear crystal to two entangled photons of the same frequency, which is one half of that of the incident photon. The fluctuations of the electric field in the squeezed light are locally lower than the vacuum fluctuations, the so-called shot-noise level. There appears to be a limit to the amount of squeezing relative to the free vacuum, which has been measured to be from -0.5 dB to the most recent value of -15 dB [10]. Detection of the squeezing relative to the free vacuum field is done using balanced homodyne detection (BHD). Marecki's QI predicts the maximum degree of squeezing in dB that is possible in terms of the fraction of the cycle during which the variance in the electric field is less than that of the free vacuum limit. Marecki developed the theoretical framework demonstrating the ability of BHD to quantify the vacuum fluctuations of the electric field in terms of vacuum expectation values of products of the electric field operators, the one- and two- point functions of arbitrary states of the electric field [11,12]. He applied this theory to the measurement of negative Casimir energy densities using BHD. The corresponding experiments require the placement of photodiodes within Casimir cavities and have yet to be performed.

To date no QI has been tested experimentally. One of the reasons for this was noted by Marecki [9]: "As far as we know quantum field theoreticians do not know that their inequalities may influence real experiments nor are quantum opticians aware of the existence of such inequalities." Most of the quantum inequalities have been developed by quantum cosmologists or quantum field theorists, who are unaware of measurements of vacuum energy done by experts in quantum optics. This paper is the first attempt to bridge this gap and test a quantum inequality with published experimental data. It appears easiest to test the QI for squeezed light because with balanced homodyne detection one measures the squeezing relative to the free vacuum, which corresponds to the theoretical quantity described in the corresponding quantum inequality. However, there may be some subtleties in the comparison because of differences in the measurement protocols. Indeed, we find that the QI as given is violated by most of the experimental data, yet all experimental data are consistent

with a theoretical model of the optical parametric amplifier (OPA) used to generate squeezed light.

2 Quantum Inequality for Squeezed Light

The QI for squeezed light gives a minimum value for the time sampled magnitude of the variance $\langle \Delta \rangle_A$ of the quantized electromagnetic field for a state A where

$$\langle \Delta \rangle_A \equiv \int_{-\infty}^{+\infty} f(t) dt (\langle E^2(x,t) \rangle_A - \langle E^2(x,t) \rangle_{vac}) \tag{3}$$

where $f(t)$ is normalized to 1.

A simplified version of Marecki's derivation is given in Appendix A. His key result is [9]

$$\langle \Delta \rangle_A \quad \geq \frac{-2}{(2\pi)^2} \int_0^{\infty} d\omega \int d^3 p \mu_p^2 \omega_p |(f^{1/2})_{FT}(\omega + \omega_p)|^2 \tag{4}$$

where $\omega_p^2 = p_1^2 + p_2^2 + p_3^2$.

The minimum value of the variance $\langle \Delta \rangle_A$ for a state A is determined by the time window function $f(t)$, specifically by $|(f^{1/2})_{FT}|^2$ the magnitude squared of the Fourier transform of the square root of the window function $f(t)$. In order to insure convergence of the integral, a spectral function $\mu_p = \mu(\omega_p - \omega_0)$, a function of ω_p that is strongly peaked at $\omega_p = \omega_0$, must be included. This term reflects the frequency response of the apparatus measuring the variance. The result Eq. 4 is similar in spirit to that of other researchers in that it involves the Fourier transform of the time window [13]. There is no proof that Eq. 4 represents the greatest lower bound for $\langle \Delta \rangle_A$.

In formulations of other Quantum Inequalities, other features of the time window, such as the second derivative, determine the minimum average energy over the time sampling [5]. In all formulations of Quantum Inequalities to date, the window function determines the minimum energy values. This may seem counter intuitive. In all cases, these formulations assume a free plane-wave electromagnetic field without sources or boundaries. We have found, like others, that the specific properties of the window function are very important [4]. Only in Marecki's calculations of the QI does a spectral function μ_p appear. This may be a problem because the Fourier transform of the time window $f(t)$ implies a certain frequency response of the apparatus, and this may conflict with the independent requirements for the function μ_p. The quantity that is generally measured in experiments is the Log_{10} of the variance for some state A relative to the variance of the free vacuum:

$$R_{expt} = 10\text{Log}_{10} \left(\frac{\langle \Delta \rangle_A + \langle E^2 \rangle_{vac}}{\langle E^2 \rangle_{vac}} \right) \tag{5}$$

According to Marecki, the measured squeezing in dB must exceed in numeric value R, where

$$R = 10\mathrm{Log}_{10}\left[\frac{\frac{1}{(2\pi)^3}\int d^3p\mu_p^2\omega_p(1-\int_0^\infty d\omega(4\pi|(f^{1/2})_{FT}(\omega+\omega_p)|^2)}{\frac{1}{(2\pi)^3}\int(\mu_p^2\omega_p)d^3p}\right] \tag{6}$$

2.1 Evaluation of R for Specific Time Sampling Functions

For a Gaussian

$$f(t) = \frac{1}{t_0\sqrt{2\pi}}e^{-t^2/2t_0^2} \tag{7}$$

we obtain

$$R = 10\mathrm{Log}_{10}\left[-\frac{\int d^3p\mu_p^2\omega_p Erf[\sqrt{2}t_o\omega_p]}{\int d^3p\mu_p^2\omega_p}\right] \tag{8}$$

If μ_p is a function of ω_p sharply peaked at ω_0, with width $\delta\omega \ll \omega_0$, then to a good approximation

$$R(\omega_o t_o) = 10\mathrm{Log}_{10}[Erf(\sqrt{2}\omega_o t_o)] \tag{9}$$

(Marecki has an additional factor of 2 in the Erf function, which we do not get). As a check on the role of the frequency windows μ_p, we can do all the integrations in R for a Gaussian frequency function, and we get an additional factor of $\omega_0^3\delta\omega$ for the ω_p integration in the numerator and in the denominator. These factors cancel, giving to lowest order in $\delta\omega$, the result quoted above. On the other hand, if we do not introduce a frequency function, we find that $R = 10\mathrm{Log}_{10}[1] = 0$, indicating that no squeezing is possible. In other words, a frequency function is required to get reasonable results; however, as we have noted the frequency function may not be consistent with the time window.

We can also compute R for a squared Lorentzian time sampling function (an ordinary Lorentzian does not give well-behaved integrals):

$$f(t) = \frac{2}{\pi}\frac{t_0^3}{(t^2+t_o^2)^2} \tag{10}$$

We find

$$R(\omega_o t_o) = 10\mathrm{Log}_{10}(1-e^{-2\omega_o t_o}) \tag{11}$$

assuming μ_p is strongly peaked at ω_o. The equation behaves similarly to the one for the Gaussian time function.

We can compute R for a square window function $f(t)$ of width ΔT with perfectly sharp corners and using a frequency function μ_p. We find that we always get perfect squeezing, $R = 10\text{Log}_{10}0$, with no dependence on ΔT. Although a perfectly sharp window is not physically possible, and is mathematically unstable, one still wonders about the meaning of this result. A sharp window allows one to do a perfect measurement (at least in principle) in which only regions of perfect squeezing are measured, and one can avoid the regions with partial or antisqueezing.

We have also evaluated the variance for a symmetric trapezoidal window with a center region T_S long and sloping sides that are each nT_s long, normalized to 1.

3 Production of Squeezed Light Using Optical Parametric Amplification

A model for an optical parametric amplifier (OPA) with balanced homodyne detection (BHD) predicts the relative variance S in the quadrature components of the vacuum electromagnetic field for a state A:

$$S = \frac{\langle E^2 \rangle_A}{\langle E^2 \rangle_{vac}} \tag{12}$$

The model [14–16] predicts that

$$S(\theta, x, \omega) = 1 + 4\beta x \left[\frac{\cos^2\theta}{(1-x)^2 + (\omega/\gamma)^2} - \frac{\sin^2\theta}{(1+x)^2 + (\omega/\gamma)^2} \right] \tag{13}$$

where $x = P/P_{th}$ is the ratio of the laser power to the power at threshold ($0 < x < 1$), β is the optical efficiency, θ is the phase difference between the local oscillator field (LO) and the vacuum field, ω is the sideband angular frequency of measurement by a spectrum analyzer, γ is the halfwidth or cavity decay rate [$\gamma = c(T+L)/l$ where $c =$ speed of light, T =transmissivity of coupling mirror, L =round trip loss, l =round trip length]. The model has been parameterized so the squeezing is a maximum at $\omega = 0$. Generally, the squeezing is given in terms of dB:

$$R = 10\text{Log}_{10} S(\theta, x, \omega) \tag{14}$$

To clarify the physical basis of the model and derive equations relating it to the QI, we briefly review the OPA model and experimental results. In recent experiments, values for the full width $2\gamma/2\pi$ range from 9 MHz to 84 MHz. Measurement frequencies ω are typically about 1 MHz to, at most, 8 MHz, $0.9 < \beta < 0.99$, and laser wavelengths vary from about 795 nm to 1064 nm. In the measurement range, (ω/γ) varies from about 0 to, at most, 1. Figure 1 shows a recent experimental arrangement [10]. A CW laser with frequency ω_{LO} encounters a polarizing beam splitter PBS, one beam going to a second harmonic generator SHG, the other beam which serves as the local oscillator LO goes to a 50–50 beam splitter in the balanced homodyne detector BHD. The beam leaving the SHG, with frequency $2\omega_{LO}$, goes to the optical parametric

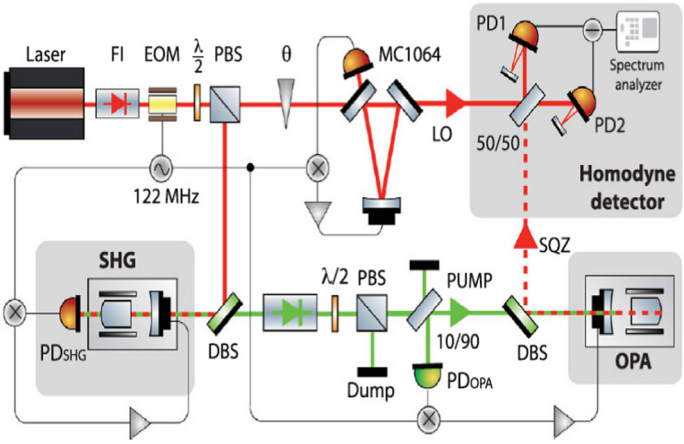

Fig. 1 Schematic of experimental setup. Squeezed vacuum states of light SQZ at a wavelength of 1064 nm were generated in a double resonant, type I optical parametric amplifier (OPA) operated below threshold. *SHG* second-harmonic generator, *PBS* polarizing beam splitter, *DBS* dichoric beam splitter, *LO* local oscillator, *PD* photodiode, *MC1064* three mirror ring cavity for spatiotemporal mode cleaning, *EOM* electro-optical modulator, *FI* Faraday isolator. The phase shifter for the relative phase θ between SQZ and LO was a piezoelectric actuated mirror [10]

amplifier OPA. The OPA is operated below threshold and is composed of a cavity with a nonlinear crystal that is fully reflective at one end and a partially reflective mirror at the other end. The OPA non-linear crystal is driven by the output of a frequency-doubled laser SHG. The crystal has a small probability of producing two photons of the same frequency ω_{LO} (half the driving frequency) by degenerate parametric down conversion. Detection is by balanced homodyne detection in which the difference in photodetector current PD1–PD2 is measured for components of the squeezed vacuum SQZ and the laser LO that have interfered at a 50–50 beam splitter. The difference current is analyzed by a spectrum analyzer, typically with a measurement bandwidth of about 100 kHz to 500 kHz.

Data on a squeezed vacuum taken from the apparatus illustrated are shown in Fig. 2 [10]. Fits based on the maximum and minimum values of S from Eq. 13 are shown in dashed lines for three power settings. In Fig. 3, the vacuum squeezing is presented as a function of the phase difference between the LO and the squeezed vacuum SQZ [10]. In this particular experiment the mirror was vibrated periodically and the abscissa given in time rather than angle, but these methods are equivalent. The first minimum would correspond to the antisqueezed quadrature $\theta = \pi/2$ radians, the second to $\pi/2$ $+\pi$ radians [17]. The fraction of the period for which the squeezing is negative equals F_T, which is an indicator of the squeezing. From measuring the graph (using curves d and a), one finds that F_T is about 0.14.

If $S(\theta, x, \omega) < 1$, then the variance or noise of this quadrature component is less than that of the free vacuum and is squeezed. This implies that the other quadrature component $R(\theta + \pi/2, x, \omega) > 1$ is antisqueezed. The minimum value of S for squeezing occurs for $\theta = \pi/2$ and equals

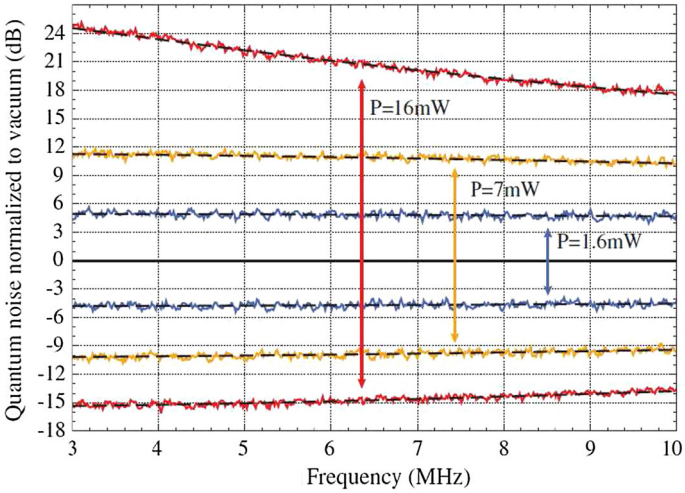

Fig. 2 Squeezing in $dB = 10\text{Log}_{10}R$ as a function of $Power = P_{th}x$, $P_{th} = 16.2\,mW$, and measurement frequency ω. Theoretical curves are shown as the narrow dashed lines with $2\gamma/2\pi = 84$ MHz and $\beta = 0.975$. The decrease in noise with increase in frequency is due to the term $(\omega/\gamma)^2$ in Eq. 13 [10]

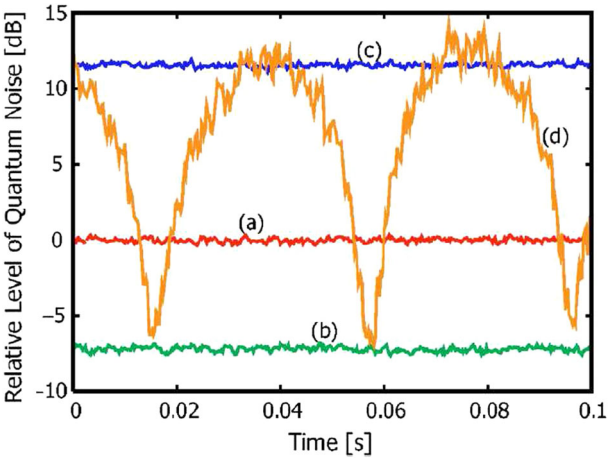

Fig. 3 Squeezing dB as a function of phase difference θ measured here by the time to move a mirror. Curve a: noise level with all inputs blanked; Curve b: phase is locked to the squeezed quadrature ($\theta = \pi/2$); Curve c: phase is locked to the antisqueezed quadrature ($\theta = 0$); Curve d: the phase is scanned. The fraction of the period that the squeezing is below zero equals F_T [10]

$$S_-(x,\omega) = 1 - \frac{4\beta x}{(1+x)^2 + (\omega/\gamma)^2} \tag{15}$$

and the maximum antisqueezing occurs for $\theta = 0$ or π and equals

$$S_+(x,\omega) = 1 + \frac{4\beta x}{(1-x)^2 + (\omega/\gamma)^2} \tag{16}$$

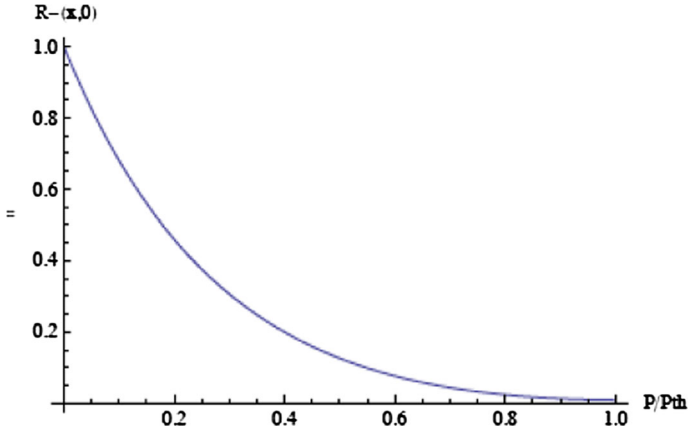

Fig. 4 The maximum squeezing $R_- = S_-(x, \omega)$ as a function of x. When $S_-(x, \omega) < 1$ there is squeezing below the normal quantum limit. We assume $\beta = 1$ and ω/γ is negligible

The maximum possible squeezing $S_-(x, 0)$ is shown as a function of x in Fig. 4. The frequency spectrum of squeezing is Lorentzian. The product of the maximum and minimum variances is

$$S_-(x, \omega) * S_+(x, \omega) = 1 - \frac{16\beta(1-\beta)x^2}{\left[(1+x)^2 + (\omega/\gamma)^2\right]\left[(1-x)^2 + (\omega/\gamma)^2\right]} \tag{17}$$

For an ideal optical system with no losses $\beta = 1$ and the product is 1, as it must be according to the Heisenberg Uncertainty Principle.

For comparison to the quantum inequality, we need to know the angular interval $\Delta\theta$ over which the light is squeezed. The light is squeezed if the term in brackets in Eq. 13 is negative, which implies

$$\frac{S_+(x, \omega) - 1}{1 - S_-(x, \omega)} < \tan^2\theta \tag{18}$$

It follows that $F_T = \Delta\theta/\pi$, which is the fraction of the period during which the light is squeezed, is given by

$$F_T(x, \omega) = 1 - \frac{2}{\pi}\tan^{-1}\sqrt{\frac{S_+(x, \omega) - 1}{1 - S_-(x, \omega)}} \tag{19}$$

For the special case of an ideal OPA, we can substitute $S_+ = 1/S_-$ to get

$$F_T(x, \omega) = 1 - \frac{2}{\pi}\tan^{-1}\sqrt{\frac{1}{S_-(x, \omega)}} \tag{20}$$

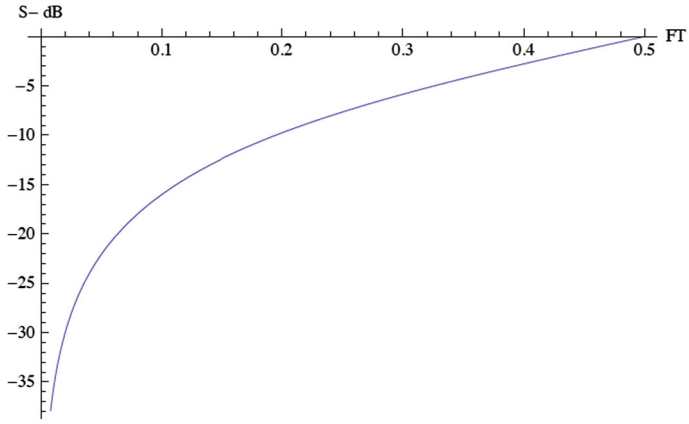

Fig. 5 $10\text{Log}_{10}S_-(x,\omega)$ versus $F_T(x,\omega)$ for an ideal OPA

which can be solved for S_- to obtain

$$S_-(x,\omega) = \tan^2\left[F_T(x,\omega)\frac{\pi}{2}\right] \tag{21}$$

which is valid for $0 < F_T < 0.5$. A plot of $R = 10\text{Log}_{10}S_-(x,\omega)$ as a function of $F_T(x,\omega)$ for an ideal OPA is shown in Fig. 5. We would not expect experimental points to display squeezing greater than the amount allowed for the ideal OPA. Since most OPA measurements were not ideal, we used the general formula Eq. 19 for $F_T(x,\omega)$ to reduce data.

The maximum fraction of time in a period during which squeezing can occur is $F_T = 1/2$, and this only occurs when x approaches zero, so the amount of squeezing is slight.

4 Analysis of Data from OPA

We analyzed data from 12 experiments conducted over the last thirty years [10,17–26]. We obtained values of F_T and squeezing/antisqueezing from plots or from the text, and estimated errors as much as possible. The most recent data is shown in Fig. 2 [10]. They fit their squeezing data (S_- and S_+) to the model, actually including a small correction for the phase uncertainty, with excellent agreement. To get F_T from their data, we used the equation from the OPA model in terms of the arctangent in Eq. 19. On the other hand, for the data in Fig. 3 we could do calculations of F_T graphically from S_- and S_+ and from $\Delta\theta$, from their plots. Thus, we can compare the two methods. (Some papers plot the squeezing vs. time shown in Fig. 2 as squeezing vs. phase difference. We treat both types of plots in the same manner with an assumed equivalence between time and phase change that corresponds to the rate at which a mirror is moved in degrees/second.) In about half the papers, we could compare the two methods and found they agreed to within about ±8% rms. When we could use

both methods to compute F_T, we used the average in our plots. For Vahlbruch we took points at three power levels (x = 0.8, 0.3, 0.1) [10]. For all other publications, we had only one power level.

5 Interpretation of Squeezing and Observation Time

To compare the results of the QI and the OPA data requires the assumption that the squeezing in the OPA analysis is equivalent to the squeezing in the QI analysis as discussed by Marecki [9]. In the OPA case, the squeezing depends on the phase difference θ between the local oscillator LO and the squeezed light SQZ, while in the QI analysis the squeezing depends on the phase change $\omega_o t_o$ occurring during the observation time. In the equation for R, which expresses the QI, t_o is the width of the time sampling function and ω_o is the center frequency in radians/sec of the frequency sampling function. From Marecki's derivation, one would assume that this corresponds to the center frequency of the laser probe. However, once we are making measurements using BHD, where the detection is based on the interference between the LO and the squeezed light, the spectrum analyzer's output at a frequency ω is a quadrature noise measurement of the optical field at frequency $\omega_{LO} + \omega$. Consequently the appropriate expression for the BHD phase change in radians during a measurement lasting a time t_o is given by the product ωt_o where ω is the BHD measurement frequency. If we observe for a time interval M, which equals the period of the squeezing $S(\theta, x, \omega)$, then the phase change is $M\omega = \pi$. Therefore $\omega t = \pi(t/M)$, where t is the observation time. Defining the fractional observation time $F_T = t/M$, we conclude that $\omega t = \pi F_T$. Thus for a Gaussian time sampling function we have

$$R(F_T) = 10\mathrm{Log}_{10}[Erf(\sqrt{2}\pi F_T)] \tag{22}$$

On the other hand, Marecki [9] just identified $\omega_o t_o = \tau$ as the fraction of the period F_T in which squeezing occurred, omitted the factor of π, and also had an additional factor of 2, thus obtaining $R(F_T) = 10\mathrm{Log}_{10}[Erf(2\sqrt{2}F_T)]$.

For the squared Lorentzian time function we obtained

$$R(F_T) = 10\mathrm{Log}_{10}(1 - e^{-2\pi F_T}) \tag{23}$$

whereas Marecki did not have the factor π in the exponent.

Note that in the derivation of the QI, ωt_o is the phase change during an observation of the variance of a quadrature component. Nothing is said in the derivation about whether the field is squeezed or not. The QI appears to place a bound on the variance for this phase change for any quadrature component, squeezed or not, during the observation time. Assuming one physically can observe the field only when it is squeezed, then we should obtain the value for $R(F_T)$ as restricted by the QI.

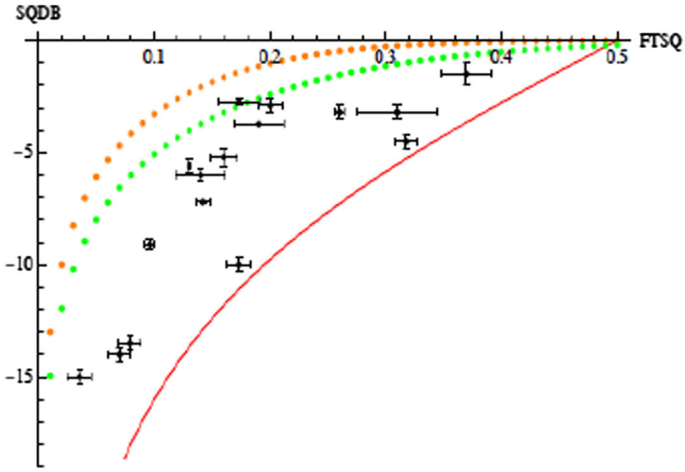

Fig. 6 Squeezing R in dB versus F_T, the fraction of observation period that is squeezed, for a Gaussian time function. The top dotted curve is derived as $10\text{Log}_{10}[Erf(\sqrt{2}\pi F_T)]$. The middle dotted curve is calculated directly from equations in Reference 9. The solid bottom curve is from the ideal OPA model. Experimental points are shown with error bars

6 Comparison of QI Predictions and OPA Data

If we assume that we are observing the variance during half of the period then $F_T = 0.5$, and the QI gives a value of $R(F_T)$ whose absolute value could not be exceeded with the maximum possible squeezing during the half period. Similarly, if we are observing for the entire period, then $F_T = 1$. By our understanding, the longer we observe, the more likely we will have regions of variance that are above the vacuum level and the bigger R will be. For the shortest times, we can have the most squeezing.

In Fig. 6 for a Gaussian time function, we have plotted our result for $R(F_T)$ (Eq. 22, top dotted curve), and Marecki's result $R(F_T) = 10\text{Log}_{10}[Erf(2\sqrt{2}F_T]$ (middle dotted curve), and $R(F_T)$ for an ideal OPA (Eq. 20, bottom solid curve). The QI is very restrictive; the degree of squeezing obtained in the experiments is greater than that allowed by either form of the QI for all but one experimental point. All data are consistent with the ideal OPA model.

The results for the squared Lorentzian time function were better than for the Gaussian, as shown in Fig. 7. Almost all points violated Eq. 23, but only about half the points violate Marecki's version of the QI with no π (middle dashed curve).

One phenomenological approach to understanding the disagreement between the data and the QI is to try reducing the argument in the equations to improve the agreement of the QI prediction with the data. As the arguments in the error function and the exponential decrease, the agreement does improve. Fitting the functional forms to the data gives the plots as shown in Fig. 8. No points violate these best fits, but the significance of them is not clear. Certainly for F_T above about 0.3, they do not appear to be sufficiently restrictive. They are not as restrictive as the ideal OPA curve.

We evaluated the variance for a symmetric trapezoidal window with a center region T_S long, and sloping sides that are each nT_S long, normalized to 1. The results (dashed

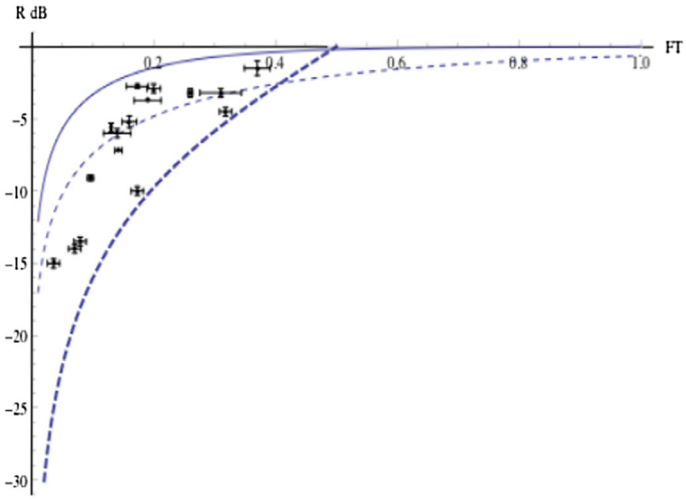

Fig. 7 R dB versus F_T for a Lorentzian squared time function. Solid line, closest to the x-axis, is our result, which includes a factor of π; middle dashed curve is Marecki's result with no π; and the thick dashed line is for an ideal OPA

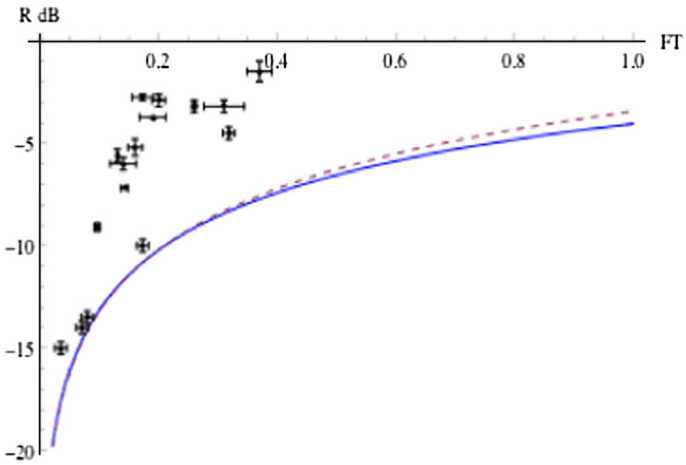

Fig. 8 R dB vesus F_T, showing the best fit for the Lorentzian (solid curve) and the Gaussian (dashed curve) time functions. For the Lorentzian, the arguments are $(1/3\pi)$ (or 1/3 for Marecki) of the theoretical values. For the Gaussian, the arguments are $(1/4\pi)$ (or 1/4 for Marecki)

curves) are displayed in Fig. 9 for a range of values of n from 0.001 (most negative black curve, and $f(t)$ is almost a square window) to 5.0 (dashed curve nearest the origin) which corresponds to a nearly triangular window. The solid curves are for the same n values, but a factor of π has been omitted in the argument in agreement with Marecki. The ideal OPA bound is the solid curve crossing all other curves. As the window becomes more triangular, the curves are less restrictive on squeezing and do not agree with the data. Only the curve for the nearly square window ($n = 0.001$)

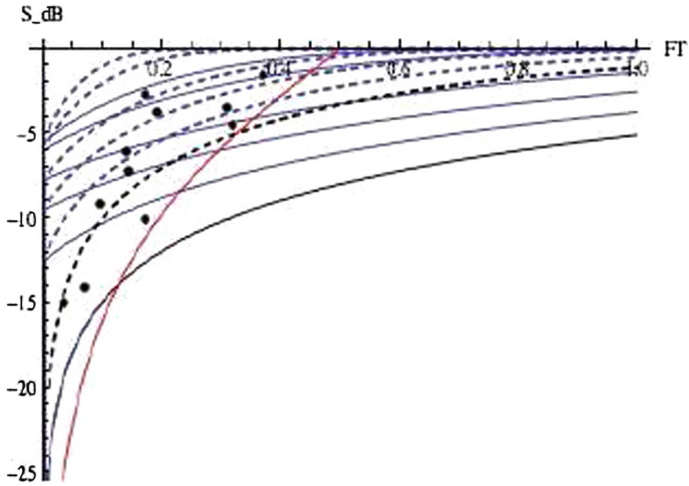

Fig. 9 $R\,dB$ $(= S_{-}\,dB)$ versus F_T for a symmetric trapezoidal time function with a center region T_S long and sides nT_S long. The dashed curves are for $n = 0.001$ (most negative R), 0.2, 0.5, 1.0, 3.0, 5.0 (nearest the origin). The solid curves are for Marecki's result, which omitted the π. The solid curve crossing the other curves is for the ideal OPA

without the π is not inconsistent with all the data. Yet this curve clearly fails to be sufficiently restrictive for values of F_T greater than about 0.3 and predicts squeezing exceeding that allowed by an ideal OPA. Mathematically this nearly rectangular window is on the edge of instability, especially for low values of the power, and as n decreases further, this window becomes a square window for which the limit is $R = 10\text{Log}_{10}0$.

Clearly the form of the time window is very important, yet for all forms examined, the resultant QI do not appear to have the right functional form or the numerical values one might expect for a QI applicable to the experimental data.

7 Discussion and Conclusion

The mathematics of the derivation of the Quantum Inequality appear sound, and the model for the OPA data has been experimentally validated, and it is consistent with all the data we examined. Yet the QI and OPA model do not appear to be consistent with each other. The QI was violated by all the data points for the Gaussian time window and about half the points for the Lorentzian squared time window. Although other windows may show improved results, these inconsistencies suggest a deeper problem.

The model for the ideal OPA gave the best results: no data exceeded its maximum squeezing, yet the most recently published data came close. It also predicted that the maximum duration of squeezed light does not exceed 1/2 the cycle, which agreed with the data, yet was not predicted by the QI. Nevertheless, the ideal OPA is a model that is an approximation with limitations.

One of the issues mentioned was the potential conflict between specifying a time function $f(t)$ and an independent frequency function $\mu(\omega)$. To explore this effect, we did a calculation assuming the Gaussian time function and a frequency function $\mu(\omega)$ which was given by the Fourier transform of the time function. Explicit calculation showed that the resulting expression for the QI was similar to that obtained without explicitly giving the precise form of the frequency window. Although this conflict is real, it does not appear to be responsible for the systemic disagreement seen between the QI and the OPA data.

Another possible issue might be the frequency dependence of the measured squeezing for the OPA data as predicted by Eq. 13. The output beam of a OPA has the Lorentzian squeezing spectrum with center frequency and halfwidth that depends on the properties of the resonant cavity. Experimental data are typically taken with a phase θ and frequency which give the maximum squeezing. Since the QI correlates the fraction of time the signal is squeezed with the dB of squeezing, we may need to account for the change in F_T for frequencies away from the sideband used for the measurement. We can compute an "effective" duration of squeezing $F_{TE}(x)$ which is weighted by integrating the frequency over the variance of the squeezed vacuum. The behavior of $F_{TE}(x)$ will depend on the range over which we integrate and the halfwidth. This integration will increase the effective size of the $F_{TE}(x)$, essentially moving all data points to the right, making the disagreement between data and the QI worse, so this is not the explanation.

Another critical issue concerns the nature of the assumed measurement in the derivation of the Quantum Inequality. The assumption is that a measurement of the energy density will be made that lasts a fraction of a cycle of oscillation of the electromagnetic field of the laser being employed. On the other hand, to make an accurate measurement using BHD requires observation for a number of cycles. It does not appear possible to make a good measurement of the squeezing if observation is for a fraction of a cycle. The measurement of the energy density in the OPA method is actually done over many cycles. For a fixed phase difference between the LO and the vacuum signal SQZ, the balanced homodyne detection automatically selects the corresponding energy output which is measured continuously over as many cycles of the laser light as desired, ensuring significant accuracy. On the other hand, no corresponding mechanism appears to be available for the measurement assumed to occur in the derivation of the QI. Thus, there may be an inconsistency between the measurement assumed in the derivation of the QI and the measurement method of the OPA. Marecki addressed this issue in an analysis of the BHD method, stressing that in the theory of the QI all operators are restricted to the frequency ω_{LO} of the LO and time was $2\pi/\omega_{LO}$ periodic and therefore $\omega_{LO}t < 2\pi$ for all times [12]. It is not clear if there is an inconsistency and if it is responsible for the disagreement between the QI and the experimental data.

The choice of window function is probably the most significant factor when applying the QI to real data. Mathematically, the choice of a window function is simple. However, when comparing theory to data, it is not clear what window function is actually appropriate for the experiment being done even though the choice dominates the restrictions due to the QI. In addition, Heisenberg and Bohr maintained that measured fields were averages over space-time volumes, whereas Marecki (Eq. A.6) and Ford (Eq. 1) only have a time average.

This work represents the only comparison to date of experimental data to the theory of a QI. Hopefully, the conundrum of the disagreement between the QI and the OPA measurements will be resolved more fully in the future with interdisciplinary collaborations and more experiments, and more detailed theoretical derivations. Our results highlight the subtleties that can be implicit in theoretical derivations of QI, particularly in the proposed measurement process. Ideally, an unambiguous experimental procedure could be associated with the theoretical derivations. These issues may also affect the applicability of the QI that have been proposed for other situations.

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Appendix A: Derivation of Marecki's Quantum Inequality

Marecki [9] derives a Quantum Inequality for squeezed light and squeezed vacuum following the general approach of Fewster and Teo [27] and Pfenning [13]. We briefly describe his derivation to clarify the comparisons to the OPA data. Marecki defines the operator variance of the normally ordered electric field $\Delta E^2(x,t)$:

$$\Delta E^2(x,t) = E^2(x,t) - \langle E^2(x,t)\rangle_{vac} \tag{A.1}$$

and considers a time sampling of the field squared

$$\Delta = \int_{-\infty}^{+\infty} dt f(t) \Delta E^2(x,t) \tag{A.2}$$

where

$$1 = \int_{-\infty}^{+\infty} dt f(t) \tag{A.3}$$

He also mentions the possibility of including a frequency sampling function $\mu_p = \mu(\omega_p - \omega_0)$, peaked at ω_0, that reflects the frequency response of the apparatus measuring the variance. Since it is necessary to use a frequency sampling function to get finite results for Δ, we will include it in our derivations. However, we note that there is a potential consistency issue using an independently selected frequency sampling function since the time sampling function $f(t)$ implies a frequency selection determined by its Fourier transform. Using the Coulomb gauge, the vector potential is

$$\begin{aligned} A_i(\mathbf{x},t) = &\frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \\ &\times \sum_{\alpha=1,2} \mathbf{e}_i^{\alpha}(\mathbf{k}) \{a_\alpha^\dagger(\mathbf{k}) e^{ikx} + a_\alpha(\mathbf{k}) e^{-ikx}\} \end{aligned} \tag{A.4}$$

where $\omega_k = |k|$ and α denotes the two polarization states which are normalized and orthogonal to $\mathbf{k}$. In the exponentials, $kx = -\mathbf{k}\cdot\mathbf{x} + \omega t$ represents the scalar product. The electric field operator is

$$E_i = -\frac{\partial A_i}{\partial t} \tag{A.5}$$

The expectation value of the time sampled free vacuum field squared is

$$\langle E^2\rangle_{vac} = \int_{-\infty}^{+\infty} dt f(t)\langle E^2(\mathbf{x},t)\rangle_{vac} \tag{A.6}$$

$$= \frac{1}{2(2\pi)^2}\int \mu_k d^3k\mu_p d^3p\sqrt{\omega_k\omega_p}2\pi f_{FT}(\omega_p-\omega_k) \times \sum_{\alpha,\beta=1,2} \mathbf{e}_i^{\alpha}(\mathbf{k})\mathbf{e}_i^{\beta}(\mathbf{p})\delta_{\alpha\beta}\delta(\mathbf{p}-\mathbf{k}) \tag{A.7}$$

where we have used the commutator $[a_\alpha(\mathbf{p}), a_\beta^\dagger(\mathbf{k})] = \delta_{\alpha\beta}\delta(\mathbf{p}-\mathbf{k})$ and included the frequency function. The Fourier transform of the time sampling function $f(t)$ is defined as

$$f_{FT}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt f(t)e^{-i\omega t} \tag{A.8}$$

Integrating Eq. A.7 over k, using the unity normalization of the polarization vectors $\sum_i \mathbf{e}_i^{\alpha}(\mathbf{k})\mathbf{e}_i^{\alpha}(\mathbf{p}) = 1$, and that $f_{FT}(0) = 1/2\pi$ because of the $f(t)$ nomalization, gives

$$\langle E^2\rangle_{vac} = \frac{1}{(2\pi)^3}\int(\mu_p^2\omega_p)d^3p \tag{A.9}$$

Substituting this result into the expression for the variance Δ gives, after integration over time,

$$\Delta = \frac{1}{2(2\pi)^2}\int d^3k d^3p\mu_k\mu_p\sqrt{\omega_k\omega_p} \times \sum_{\alpha,\beta=1,2} \mathbf{e}_i^{\alpha}(\mathbf{k})\mathbf{e}_i^{\beta}(\mathbf{p})\{a_\alpha^\dagger(\mathbf{k})a_\beta(\mathbf{p})e^{i(-\mathbf{k}+\mathbf{p})\mathbf{x}}f_{FT}(\omega_p-\omega_k) - a_\alpha(\mathbf{k})a_\beta(\mathbf{p})e^{i(\mathbf{k}+\mathbf{p})\mathbf{x}}f_{FT}(\omega_p+\omega_k) + HC\} \tag{A.10}$$

where HC is the Hermitian conjugate. To derive a quantum inequality, Marecki defines a vector operator $B_i(\omega)$ and computes the integral over frequency of the norm of $\mathbf{B}$ which has to be positive

$$\int_0^{\infty} d\omega B_i^\dagger(\omega)B_i(\omega) > 0 \tag{A.11}$$

We choose

$$B_i(\omega) = \frac{1}{\sqrt{2\pi^2}} \int d^3p \sqrt{\omega_p} \times \sum_{\alpha,\beta=1,2} \mathbf{e}_i^{\alpha}(\mathbf{p}) \{ a_\alpha(p) (f^{1/2})^*_{FT}(\omega - \omega_p) e^{i\mathbf{px}} - a_\alpha^\dagger(p) (f^{1/2})^*_{FT}(\omega + \omega_p) e^{-i\mathbf{px}} \} \tag{A.12}$$

and substitute this into Eq. A.11, and use the result in Eq. A.10. After taking the expectation value with respect to state A, we obtain Eq. 4 in Sect. 2. Note that in Eq. A.12, $(f^{1/2})^*_{FT}(\omega - \omega_p)$ means the complex conjugate of the Fourier transform of the square root of $f(t)$.

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